Optimal Acquisition Quantities in Remanufacturing with Condition Uncertainty

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The condition of the used products acquired by remanufacturing firms often varies widely. A firm can manage this variation by acquiring a quantity of used items that exceeds demand, enabling it to remanufacture a subset of the acquired items in the best condition. As more excess items are acquired, the firm can increase its selectivity and lower its remanufacturing costs. In this paper, we examine the tradeoff of acquisition and scrapping costs versus remanufacturing costs when used product condition is widely varying and uncertain. We derive acquisition quantities that minimize total expected costs for several representations of condition variability and remanufacturing cost structures. We find that, when costs are linear, the optimal acquisition quantity has a closed form and increases with the square root of the degree of condition variability. Our models are based on experience with remanufacturers of cell phones and imaging supplies, and application of our results is illustrated using example data from industry.

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1. Introduction

The inherent variation in the condition of used products presents a significant challenge for remanufacturing firms. Faced with a given demand, a remanufacturer can manage this variation and the cost of remanufacturing through the quantity of used items it acquires externally. As the number of excess items acquired is increased, the firm can be more selective and remanufacture only those items that are in the best condition. Thus, acquisition quantity is a key managerial decision for remanufacturers that has a direct impact on unit remanufacturing costs. The determination of an optimal acquisition quantity is fairly straightforward when the distribution of item conditions within any lot is known with certainty before the lot size is determined. However, there is likely to be some uncertainty regarding the actual conditions of the used products to be acquired. In this research we derive optimal acquisition quantities under condition uncertainty for several remanufacturing contexts. Our work is related both to the growing research area of remanufacturing and closed-loop supply chains as well as lot sizing with uncertainty. Below, we briefly review the literature in these areas and define our contribution to each.

The management of remanufacturing has received considerable attention from researchers in recent years. For a thorough review of the academic work in this area, we refer the reader to Souza (2008) and Atasu et al. (2008). While the variability of used product condition has been well documented (Bloemhof-Ruwaard et al. 1999, Guide and Jayaraman 2000, Fleischmann et al. 2000, Toktay et al. 2000), the impact of condition variability on used product acquisition decisions – an area identified by Guide et al. (2003) as under-treated from an academic perspective – has only recently begun to receive attention from the research community. Aras et al. (2004) was the first analytical study to explicitly model quality categorization of used items, using the context of a hybrid manufacturing/remanufacturing firm. Several models have examined the acquisition decision when used product condition variability is fully captured by two categories – remanfacturable or not (Zikopoulos and Tagaras 2007, 2008). Galbreth and Blackburn (2006) consider a range of remanufacturable conditions, but their model assumes that the condition distribution of an acquired lot is known with certainty. In this paper, we model acquisition decisions when multiple remanufacturable conditions are possible and the condition of each acquired item is uncertain. Our models provide results that are relevant for remanufacturing practice, where condition is likely to be both widely varying and uncertain.

In deriving optimal acquisition quantities, our work is related to the literature on optimal lot sizes for a single production run when there are random reject rates (e.g. Levitan 1960, Wein 1992, Grosfield-Nir and Gerchak 1996, Nandakumar and Rummel 1998). For detailed discussions and reviews of lot sizing models of this type, we refer the reader to Yano and Lee (1995) and Grosfield-Nir and Gerchak (2004). The important difference between our work and these models is that, in remanufacturing, the randomness occurs in the condition of the raw materials (used items), not the production process. Since used items can be inspected at the time of acquisition and processed in order of condition, the average unit remanufacuring cost of the items processed to meet a given demand is a function of the quantity of used items acquired, with higher acquisition quantities yielding a lower expected remanufacturing cost. This remanufacturing-specific context requires a new formulation of the lot sizing problem.

The remainder of this paper is organized as follows. In Section 2 we present an overview of our model and derive a simple baseline solution for the case where product condition is defined by a continuum. In Section 3 we add condition uncertainty to this model, deriving results for both linear and nonlinear cost functions. Section 4 examines the case where condition is best captured using discrete categories as opposed to a continuous function. We conclude in Section 5 with a discussion of key results and the sensitivity of the model to parametric changes.

2. Model Overview

We motivate our analysis using our experiences with CertiCell LLC, an independent cell phone remanufacturer. Although we use this firm as our data source, our models and results are applicable across a variety of other remanufacturing contexts, including large segments of the remanufacturing industry such as toner cartridges and power tools (toner cartridges alone account for around \$2.5B in annual revenues (Hauser and Lund 2003)). As with many remanufacturable goods, an active collection/broker community exists for used cell phones, and independent firms can acquire used items on the open market as needed, allowing timing and quantity of used item inflow to be fully controlled. In this paper we develop models that capture the tradeoff between acquisition quantity and remanufacturing cost at this type of firm, and the motivating example of CertiCell provides us with data to which these models can be applied. Specifically, CertiCell provided data on actual acquisition costs, as well as estimated remanufacturing costs for various phone models processed (Elliott 2008).

Condition of used items can be highly variable – a used cell phone could have any combination of a wide array of potential remanufacturing needs (antenna, screen, microphone, speaker, faceplate, etc.). In some cases, these conditions can be captured with a few discrete condition categories. In other cases, the combinations of remanufacturing needs are so numerous that the range of possible conditions, while technically a discrete set, is closely approximated by a continuum. Our initial focus in this paper is on the continuous condition case, and we develop a model for discrete condition categories in Section 4.

Our models are applicable for firms that remanufacture to order – demand takes the form of a specific, known production target (or, equivalently, there is no forecast error). Examples of this context include a toner cartridge remanufacturer that produces private label cartridges based on orders from an office supply store or a cell phone remanufacturer that fulfills specific orders from a phone insurance provider. For many remanufactured goods, especially electronic items, prices fall rapidly, and the high rate of obsolescence means that there is little interest in holding inventory of remanufacturable products, nor is there a guarantee that more used items will be available at a later date. Thus, items are rarely remanufactured to stock, and we can realistically limit our analysis to a single-period model. We also assume that shortages are disallowed, reflecting the situation where a firm loses considerable goodwill if demand is not satisfied in full (and, if supplying a large retail operation, risks being eliminated as a preferred supplier).

We consider the problem of a remanufacturer who, facing a single demand D, must decide how many used items to acquire. For example, a firm might face a demand for 500 remanufactured cell phones. To avoid shortages, at least 500 used phones must be acquired. Given condition variability, acquiring additional phones beyond the 500 needed enables the remanufacturer to meet demand without having to remanufacture those items that are in the worst condition. Similar to Ferguson et al. (2009), we model the condition λ of each used item as a real number $\lambda \in [0, 1]$, the value of which is a random variable with cumulative distribution $G(\lambda)$ and density function $g(\lambda)$. In our model, $\lambda = 0$ represents the best possible condition and $\lambda = 1$ the worst possible condition (remanufacturing cost is an increasing function of λ). The decision variable for the remanufacturer is the quantity of used items acquired Q, where $Q \geq D$. All Q items are inspected and rank ordered by condition, allowing the D items in the best condition to be remanufactured. Unit acquisition and inspection costs are denoted u, and unit scrap cost, which captures both the direct and indirect (environmental) costs of scrapping, is denoted s. We assume that all acquired items

Table 1: Notation Summary

Parameters	
D	demand for remanufactured items
λ	used item condition
$G(\lambda)$	condition cdf
k	index for ordered item condition within an acquired lot
	($k = 1 \rightarrow$ best condition; $k = Q \rightarrow$ worst condition)
u	unit acquisition and inspection costs
s	unit scrap cost
a	fixed portion of unit remanufacturing cost
c	variable (condition-dependent) portion of unit remanufacturing cost
Decision Variable	
Q	quantity of used items acquired

are inspected so that they can be processed according to their unique remanufacturing needs. We also assume perfect testing and no capacity constraints (similar assumptions are made in both Guide et al. (2003) and Galbreth and Blackburn (2006)). Our notation is summarized in Table 1.

Remanufacturers are frequently unable to obtain specific information regarding the exact distribution of the condition of used items prior to acquisition, but condition typically varies widely and is not generally characterized by a specific distribution. A diffuse distribution such as the uniform captures the high degree of condition uncertainty while providing analytical tractability. In this paper, we assume that $g(\lambda)$ is a uniform distribution. Uniformly distributed condition has also been assumed in other remanufacturing models (Imtanavanich and Gupta 2005).

To establish a baseline solution, we make two simplifying assumptions, both of which are relaxed in subsequent sections. First, we assume that, in each acquired lot, item condition is distributed exactly according to $g(\lambda)$. In effect, this assumption removes all uncertainty regarding the condition of acquired items, capturing only the variability of condition. Given this assumption, when Q items are acquired, the conditions of the best D items are uniformly distributed over $[0, \frac{D}{Q}]$. Second, we assume that remanufacturing cost is a linear function of condition $a + c\lambda$, where a is a known, fixed unit cost (for disassembly, cleaning, etc.) and c is a variable component that depends on condition. The parameter c represents the difference in remanufacturing cost between the best and worst condition items, i.e. the degree of condition variability. This simplified model is equivalent to the linear acquisition cost model in Galbreth and Blackburn (2006), which we examine for the special case of uniformly distributed condition to obtain a closed form solution. Total acquisition and remanufacturing costs are as follows:

$$f\left(Q\right)=uQ+\left(Q-D\right)s+aD+\frac{cD^{2}}{2Q} \tag{1}$$

(1) is convex in Q, with first derivative:

$$f'\left(Q\right)=u+s-\frac{cD^{2}}{2Q^{2}} \tag{2}$$

Given (2), we have the following Observation, where the optimal value of Q is rounded to the nearest integer by adding 0.5 and taking the floor function:

Observation 1 *When the condition distribution of used items is known with certainty and remanufacturing cost is a linear function of condition, the cost-minimizing acquisition quantity Q^* is:*

$$Q^{*}=MAX\left\{D,\left\lfloor D\sqrt{\frac{c}{2\left(u+s\right)}}+0.5\right\rfloor\right\} \tag{3}$$

Returning to our example of a cell phone remanufacturer, consider the acquisition decision given the following cost parameters, which are based on values from CertiCell. For one phone model, the cost to acquire and inspect each unit is \$3.00, and the variation in remanufacturing cost between the best and worst conditions is \$8.00. Scrapping costs are negligible. We can evaluate (3) at these parameter values ($u=3$, $c=8$, $s=0$) for a hypothetical demand of $D=500$, giving an optimal acquisition quantity $Q^{*}=577$. Thus, the firm minimizes the cost to meet the demand for 500 remanufactured phones by acquiring 577 used phones, inspecting them, and processing the 500 phones in the best condition.

The analysis above assumed a very simple situation where, although condition is variable, its distribution is known with certainty. In the remainder of our analysis, this assumption is relaxed.

3. The Impact of Condition Uncertainty

In this section, we explicitly consider the condition of each item to be a random variable, uniformly distributed on $[0,1]$. Consider an acquisition lot of Q used items. Let $X_{(1)},\ldots,X_{(Q)}$ denote the order statistics of the Q items, ordered by condition. Thus, the remanufacturing cost of the best D items is given by $\sum_{k=1}^{D}\left(a + cX_{(k)}\right)$. For the uniform distribution, the k^{th} order statistic, $X_{(k)}$, has the following density function for λ:

$$Q\left(\begin{array}{c} Q-1 \\ k-1 \end{array}\right)\lambda^{k-1}(1-\lambda)^{Q-k}; \quad \lambda \in [0,1]$$

We rewrite the expected cost function (1) to reflect condition uncertainty as follows:

$$f(Q) = uQ + s(Q-D) + \sum_{k=1}^{D}\left[a + \int_{0}^{1} Q\left(\begin{array}{c} Q-1 \\ k-1 \end{array}\right)(c\lambda)\,\lambda^{k-1}(1-\lambda)^{Q-k}\,d\lambda\right] \tag{4}$$

And the optimal acquisition quantity is defined by the following proposition:

Proposition 1 *When the condition is uncertain and remanufacturing cost is a linear function of condition, the cost-minimizing acquisition quantity Q^* is:*

$$Q^* = MAX\left\{D, \left\lfloor\sqrt{\frac{cD(D+1)}{2(u+s)}} - 0.5\right\rfloor\right\} \tag{5}$$

Proof: See Appendix

From (5) it is clear that an increase in D will result in a proportional increase in Q^*. In (5) we also see that the fixed portion of remanufacturing cost a is not relevant to the acquisition quantity decision. Since a change in scrap value is equivalent to a change in acquisition cost, we can evaluate the sensitivity of the solution to changes in $(u+s)$ or c. The optimal acquisition quantity Q^* increases with the degree of condition variability c. The intuition for this result is that as variability increases the selectivity enabled by acquiring extra items has increasing value. Less intuitive is the magnitude of this relationship. Observe that, for a given demand, Q^* increases with the square root of c. Similarly, Q^* decreases with the square root of $(u+s)$. These are useful guidelines for how acquisition quantities should be adjusted in response to changes in the degree of condition variability, acquisition, or scrap costs. For example, if the range of remanufacturing costs, c, doubles from 8 to 16, then the amount acquired would increase by $\sqrt{2}$. If $(u+s)$ doubles from 3 to 6, the amount acquired decreases by $\sqrt{2}$.

From the similarity between (5) and (3), we note that, given the assumptions of uniform condition and linear remanufacturing costs, the optimal acquisition quantity is closely approximated by the formulation that ignores condition uncertainty, particularly for larger demands. Recall that (3) assumes that the conditions of the acquired items are distributed exactly according to expectations. As suggested by Galbreth and Blackburn (2006), this is a reasonable assumption for larger acquisition quantities. Even for smaller demands, the penalty for ignoring condition uncertainty is minimal. In our cell phone example ($D = 500$, $u = 3$, $c = 8$, s=0), we find using (5) that the optimal value $Q^* = 577$ is identical to the one specified by the deterministic model in (3).

3.1 Results for Nonlinear Cost Functions

Of course, remanufacturing costs may not always be a linear function of condition. As pointed out by Ferguson et al. (2009), the exact expression for cost as a function of condition is difficult to estimate. In that paper the authors represent remanufacturing cost as a general nonlinear power function of condition. We make a similar assumption, modeling remanufacturing cost as $a + c\lambda^{\beta}$, where β is the shape parameter of the cost curve. Since costs increase as condition deteriorates, $\beta > 0$. Remanufacturing cost is convex in condition for $\beta > 1$ and concave in condition for $\beta < 1$. A power function with $\beta > 1$ is often appropriate since marginal remanufacturing costs are likely to increase as condition deteriorates. We illustrate with a quadratic remanufacturing cost function, i.e. remanufacturing cost equals $a + c\lambda^2$. In this case, the expected total cost is:

$$f(Q) = uQ + s(Q - D) + \sum_{k=1}^{D} \left[a + \int_0^1 Q \begin{pmatrix} Q-1 \\ k-1 \end{pmatrix} \left(c\lambda^2\right) \lambda^{k-1} (1-\lambda)^{Q-k} \, d\lambda \right] \tag{6}$$

And we have the following proposition:

Proposition 2 *When remanufacturing cost is a quadratic function of condition, the total cost function is convex in Q, and Q^* is the maximum of D and nearest integer-valued solution to the following:*

$$\frac{(Q+1)^2 (Q+2)^2}{2Q+3} = \frac{cD(D+1)(D+2)}{3(u+s)} \tag{7}$$

Proof: See Appendix

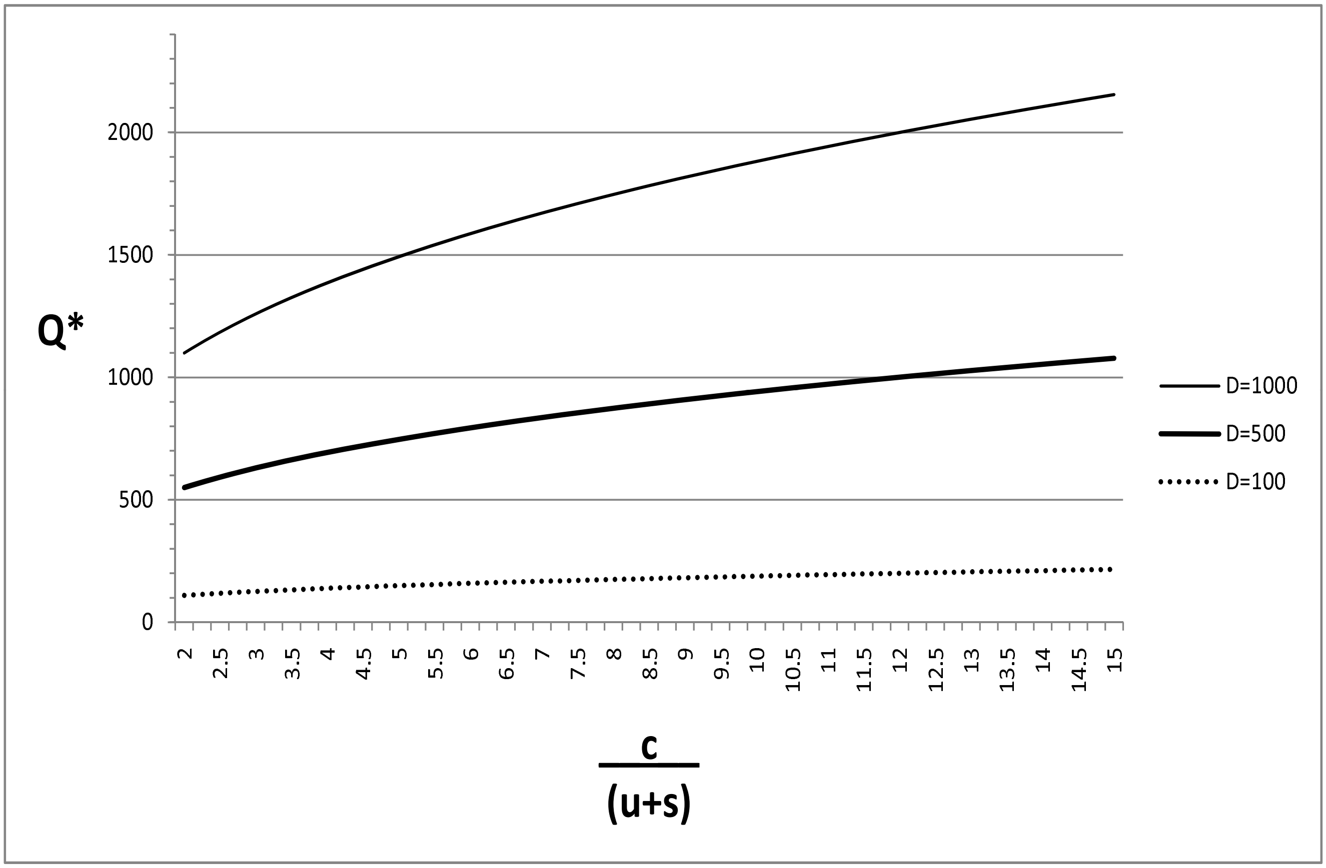

Figure 1: Sensitivity of Q^* to parametric changes

The solution to (7) can be computed using known algorithms for quartic functions. It can also easily be found numerically, beginning with $Q = D$ and incrementing Q until the best integer solution is found. Returning to the cell phone example of the previous sections, if marginal remanufacturing costs increase with condition, making this cost curve a more accurate reflection of reality, then with ($D = 500$, $u = 3$, $c = 8$, s=0) the optimal acquisition quantity can be calculated as $Q^* = 605$.

In the case of quadratic costs, Q^* is clearly increasing in D. For a given D value, Q^* appears to be a concave increasing function of $\frac{c}{u+s}$, the ratio of the remanufacturing cost range to the acquisition plus scrap cost (see Figure 1). In fact, for large D, (7) is closely approximated by the expression $\frac{1}{2}Q^3 = \frac{1}{3}\frac{c}{(u+s)}D^3$, leading to the following observation:

Observation 2 *When remanufacturing cost is a quadratic function of condition, for large values of D the ratio $\frac{Q^*}{D}$ is approximated by $\sqrt[3]{\frac{2}{3}\frac{c}{(u+s)}}$.*

4. A Discrete Condition Model

In this section we consider the case where condition variability can be captured by a discrete set of condition categories. For example, for some cell phones and other electronics, a single component (in CertiCell's case, often the cell phone's LCD screen) may account for such a large portion of the cost variation that a two-category sorting system suffices. A similar system is appropriate for some simple toner cartridge models as well, where condition is defined based on whether the item has been previously remanufactured. In addition to being common in practice, a two-category system has been studied in prior research (Aras et al. 2004, Zikopoulos and Tagaras 2007, Zikopoulos and Tagaras 2008). We also focus on the case of two condition categories, making our model similar to the one examined by Zikopoulos and Tagaras (2007), except that we allow for the remanufacturing of both categories of items, at different costs ("low cost" and "high cost").

Given two condition categories for remanufacturable items, in which each item has a probability α of being low cost (independent of the others), the number of low cost items in an acquired batch is binomially distributed. In a traditional manufacturing setting, the binomial distribution is appropriate in situations where the production of any individual item is independent of all other items (Yano and Lee 1995, Barad and Braha 1996, Grosfeld-Nir and Gerchak 2004). This is also the case in remanufacturing, since remanufactured items are collected in lots that typically have little relationship to their original manufacturing sequence. Karaer and Lee (2007) established that the binomial distribution is a valid representation of uncertainty in the fraction of used items that fall into different condition categories.

By taking $\{u, \alpha\}$ as exogenous, we do not explicitly model the case in which α is a function of the acquisition price, as in Guide et al. (2003). This reflects our experience with firms that acquire ungraded lots of items from brokers. However, if graded items are offered at different prices, a remanufacturer can use our model to select from a "market basket" of price/condition pairs $\{u_i, \alpha_i\}, i = 1...n$. In these cases, the remanufacturer can solve our model for each pair in turn and choose the pair(s) that would enable it to meet demand at the lowest total cost.

When condition is a continuous random variable as assumed in previous sections, items are ordered by condition during inspection. In the dichotomous condition context, the sorting process is more straightforward – items are simply separated into two categories, low cost and high cost. The remanufacturer will process low cost items first. If demand cannot be met

Table 2: Additional Notation for Discrete Condition

α	expected proportion of used items in the low cost category
N	actual number of used items in the low cost category
c_1	cost to remanufacture low cost items
c_2	cost to remanufacture high cost items
$\tilde{s}$	incremental remanufacturing cost of a high cost item

using only low cost items, then high cost items are processed until demand has been fulfilled. We assume that low cost and high cost items have known, fixed costs to remanufacture of c_1 and c_2, respectively. Let $\tilde{s} = c_2 - c_1$, that is, $\tilde{s}$ is the additional unit remanufacturing cost incurred for those units of demand not met using low cost items. The additional notation required for this discrete condition model is summarized in Table 2.

If N is the number of low cost items in an acquired batch of size Q, then we have the following expected cost expression when N is a binomially distributed random variable:

$$f(Q) = uQ + (Q - D)s + c_1 D + \tilde{s} \sum_{N=0}^{D-1} \binom{Q}{N} \alpha^N (1-\alpha)^{Q-N} [D - N] \tag{8}$$

Below we show that the expression (8) is "discrete convex," i.e. first differences of the function are monotonically increasing in the decision variable (Barad and Braha 1996), and therefore a unique global minimizer exists for (8).

Proposition 3 *The cost function (8) is discrete convex in Q.*

Proof: See Appendix

Given Proposition 3, the unique minimizer of (8) defines Q^* for any two-condition remanufacturing problem.

We return to the cell phone remanufacturing example to demonstrate how (8) can be used in practice. Disguised cost data for a CertiCell phone whose condition variability can be captured by two categories is as follows: $u = 3.5$, $s = 0$, $\alpha = 0.9$, $c_1 = 10$, $c_2 = 16$. In this case, the acquisition quantity that would enable the remanufacturer to meet the demand for 500 remanufactured phones at the minimal expected cost can be found by solving (8) for these parameters, giving $Q^* = 552$. Clearly, the acquisition quantity will decrease as α increases, as a higher α enables the same expected number of low cost items to be obtained from a smaller lot.

Next, we note that the case where $Q^* = D$ is defined by a simple condition:

Observation 3 *When condition is dichotomous, $Q^* = D$ whenever $u \geq \alpha\tilde{s} - (1-\alpha)\, s$.*

Observation 3 can be confirmed by noting that the expected benefit from acquiring unit $D+1$ is less than the acquisition plus expected scrap cost whenever $\alpha\tilde{s} \leq u + (1-\alpha)\, s$.

We can also define a simple inflection point, $\frac{D}{\alpha}$, in the optimal solution, as follows:

Proposition 4 *For sufficiently large D, if $u < \alpha\tilde{s}$ and $s = 0$, then $\frac{D}{\alpha}$ is a bound on the optimal solution as follows: whenever $u \leq \frac{\alpha\tilde{s}}{2}$, $\frac{D}{\alpha}$ is a lower bound on Q^*; whenever $u > \frac{\alpha\tilde{s}}{2}$, $\frac{D}{\alpha}$ is an upper bound on Q^**

Proof: See Appendix

Since first differences of the cost function are given by $\Delta f(Q) = u + s - \tilde{s}\alpha + \tilde{s}\alpha\,(\Psi(Q))$ (See proof of Proposition 3), we know that if the remanufacturing costs of both condition categories change by the same amount (i.e. $\tilde{s}$ is constant), Q^* is invariant. In terms of remanufacturing costs, only a change in $\tilde{s}$ affects Q^*. In addition, the value of $\alpha\tilde{s}$ relative to the acquisition cost u drives a simple optimality condition for Q^* ($\alpha\tilde{s} = u$, from Observation 3 without scrapping costs) as well as an inflection point in the optimal solution ($\alpha\tilde{s} = 2u$, from Proposition 4). Figure 2 illustrates the impact of $\alpha\tilde{s}$ on Q^*.

As seen in Table 3, the primary effects of $\tilde{s}$ and α on the optimal solution are defined by Observation 3: Q^* simply equals D whenever the inequality in that observation holds. When the inequality in Observation 3 does not hold, the effects of changes in these two parameters are quite different: Q^* is very sensitive to changes in α and only weakly affected by changes in $\tilde{s}$. This can be explained by the fact that, in these cases, remanufacturing of high cost items is sufficiently expensive that these items will be used primarily as safety stock against a shortfall of low cost items. Since very few high cost items are expected to be remanufactured, their exact cost premium ($\tilde{s}$) has a minor impact. For example, when $\alpha = 0.7$, a doubling of $\tilde{s}$ from 6 to 12 would change Q^* by less than 3% (from 698 to 718). On the other hand, when $\tilde{s} = 12$, a doubling of α from 0.3 to 0.6 would reduce Q^* significantly (from 1552 to 835). Essentially, roughly $\frac{D}{\alpha}$ used items will be acquired, so Q^* is roughly inversely proportional to changes in α—e.g., if α doubles Q^* is reduced by about half. Note that Proposition 4 can also be confirmed to hold in the numerical analysis presented in Table 3. Specifically, αQ^* will be greater than D whenever acquisition cost is sufficiently small ($u \leq \frac{\alpha\tilde{s}}{2}$), a situation that can be thought of as acquiring a positive safety stock of high cost items. Otherwise, αQ^* will be less than D, i.e. a negative safety stock is acquired.

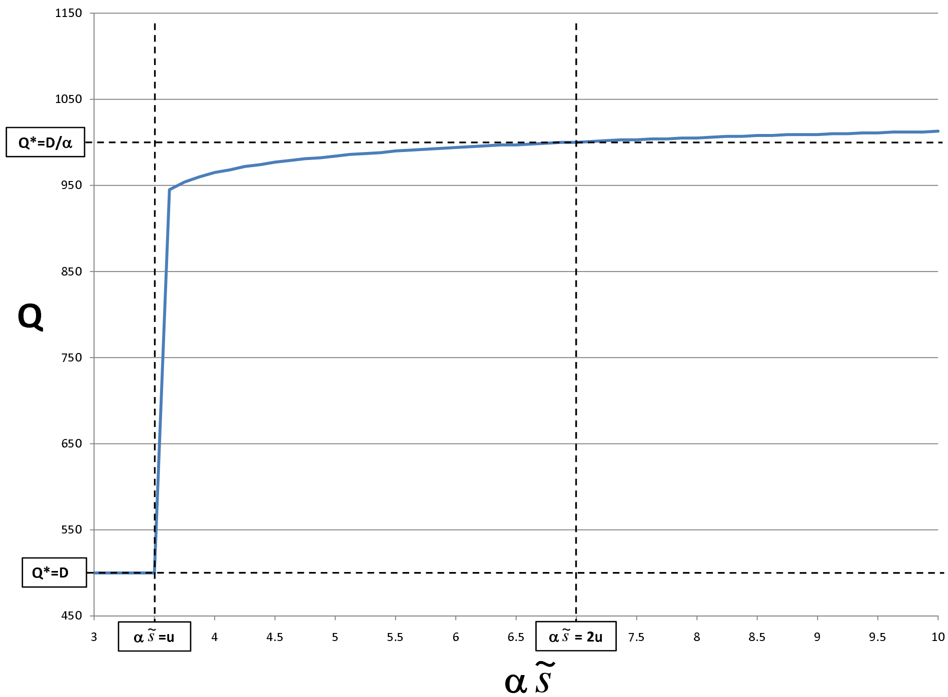

Figure 2: Q^* for various $\alpha\tilde{s}$ (D=500, u=3.5, s=0, α=0.5)

5. Conclusions

In this paper, we analyze the acquisition lot sizing problem with condition uncertainty in remanufacturing. We present results when condition variability is described by a continuum as well as by two discrete categories of remanufacturable used items. For continuous condition, we consider remanufacturing costs that are linear and nonlinear functions of condition. In each case, we show that a unique optimal acquisition quantity exists and can be found by a simple computation. We also find that the model with linear costs has a closed form solution that is very similar to that of a simple model that ignores condition uncertainty. From this closed form result we observe that the optimal acquisition quantity increases with the square root of the degree of condition variability. In addition to being of interest to remanufacturing researchers and practitioners, our results extend the knowledge of lot sizing with defects to the remanufacturing context, where each unit of raw material has its own (uncertain) processing cost.

There are several ways in which this research stream can be extended. While the assumption of uniformly distributed condition allowed us to capture significant condition variability while maintaining tractability, other distributions could also be examined. While optimal solutions may be difficult to find in these cases, methods such as simulation would enable

Table 3: Q^* for various α, $\tilde{s}$ ($D = 500$, $u = 3.5$, s=0)

	$\tilde{s}$							
α	6	8	10	12	14	16	18	20
0.2	**500**	**500**	**500**	**500**	**500**	**500**	2316	2388
0.3	**500**	**500**	**500**	1552	1608	1630	1644	1654
0.4	**500**	**500**	1202	1224	1237	1245	1252	1257
0.5	**500**	965	984	994	1000	1005	1009	1013
0.6	790	820	829	835	839	842	845	847
0.7	698	709	715	718	721	723	725	727
0.8	618	624	627	630	632	633	634	635

the investigation of a broad set of possible condition distributions and remanufacturing cost functions in terms of their impact on acquisition policies. Another interesting extension of this work would be to incorporate the potential for inaccurate sorting, as described by Zikopoulos and Tagaras (2007). Finally, the model could be extended to a multi-period setting, which might be appropriate for some remanufactured items with longer life cycles.

Appendix: Proofs

Proof of Proposition 1

Moving constant terms outside of the integral in (4) gives the following:

$$f\left(Q\right) = uQ + s\left(Q - D\right) + \sum_{k=1}^{D}\left[a + cQ\begin{pmatrix} Q-1 \\ k-1 \end{pmatrix}\int_{0}^{1}\left(\lambda\right)\lambda^{k-1}\left(1-\lambda\right)^{Q-k}d\lambda\right] \tag{9}$$

To evaluate (9) we use the following standard integral form, which can be verified by differentiation:

$$\int \lambda^m \left(a + b\lambda\right)^n d\lambda = \frac{\lambda^{m+1}\left(a+b\lambda\right)^n}{m+n+1} + \frac{an}{m+n+1}\int \lambda^m \left(a+b\lambda\right)^{n-1} d\lambda \tag{10}$$

Using (10), the integral in (9) can be expanded and evaluated as:

$$\int_{0}^{1}\lambda^k\left(1-\lambda\right)^{Q-k}d\lambda = \left.\frac{\lambda^{k+1}\left(1-\lambda\right)^{Q-k}}{Q+1}\right|_{0}^{1} + \frac{Q-k}{Q+1}\int_{0}^{1}\lambda^k\left(1-\lambda\right)^{Q-k-1}d\lambda$$

which reduces to:

$$\int_{0}^{1}\lambda^k\left(1-\lambda\right)^{Q-k}d\lambda = \frac{Q-k}{Q+1}\int_{0}^{1}\lambda^k\left(1-\lambda\right)^{Q-k-1}d\lambda$$

since the first term vanishes. Repeated application of (10) until the exponent of $(1-\lambda)$ is reduced to zero yields:

$$\frac{(Q-k)(Q-k-1)\dots(1)}{(Q+1)(Q)\dots(k+2)}\int_0^1 \lambda^k d\lambda \quad = \quad \left(\frac{(Q-k)!\,(k+1)!}{(Q+1)!}\right)\left(\frac{1}{k+1}\right)$$

Then

$$Q\left(\begin{array}{c} Q-1 \\ k-1 \end{array}\right)\int_0^1 \lambda^k (1-\lambda)^{Q-k}\, d\lambda = Q\left(\begin{array}{c} Q-1 \\ k-1 \end{array}\right)\left(\frac{(Q-k)!\,(k+1)!}{(Q+1)!}\right)\left(\frac{1}{k+1}\right) = \frac{k}{Q+1}$$

Substituting back into (9), we have:

$$f(Q) = uQ + s(Q-D) + \sum_{k=1}^{D}\left[a + \frac{ck}{Q+1}\right] = uQ + s(Q-D) + aD + c\frac{(D)(D+1)}{2(Q+1)} \quad (11)$$

which is convex in Q. The first derivative of (11) with respect to Q is:

$$u + s - c\frac{(D)(D+1)}{2(Q+1)^2}$$

and the optimal Q is:

$$Q^* = \sqrt{\frac{cD(D+1)}{2(u+s)}} - 1 \quad (12)$$

Since Q must be integer, (12) is rounded to the nearest integer by adding 0.5 and taking the floor function:

$$Q^* = \left\lfloor \sqrt{\frac{cD(D+1)}{2(u+s)}} - 0.5 \right\rfloor$$

For completeness, we add our assumption that $Q \geq D$.

Proof of Proposition 2

Using the same approach as in the proof of Proposition 1, we find that the expected remanufacturing cost for D items when Q are acquired ($Q \geq D$) is:

$$aD + c\sum_{k=1}^{D}\frac{(k+k^2)}{(Q+1)(Q+2)} \quad (13)$$

We can remove the summation from (13) by substituting an equivalent expression for the finite series in the numerator, giving the following:

$$aD + \frac{cD\,(D+1)\,(2D+4)}{6\,(Q+1)\,(Q+2)}$$

Total expected cost is therefore:

$$f\,(Q) = uQ + s\,(Q-D) + aD + \frac{cD\,(D+1)\,(2D+4)}{6\,(Q+1)\,(Q+2)}$$

which is convex in Q and has the following derivative:

$$f'\,(Q) = u + s - \frac{cD\,(D+1)\,(2D+4)\,(2Q+3)}{6\,(Q+1)^2\,(Q+2)^2} \tag{14}$$

Setting (14) equal to zero yields:

$$\frac{(Q+1)^2\,(Q+2)^2}{2Q+3} = \frac{cD\,(D+1)\,(D+2)}{3\,(u+s)}$$

Proof of Proposition 3

Note: this proof follows the approach used by Barad and Braha (1996).

For any Q, we have the following function for $f(Q+1)$:

$$f\,(Q+1) = u(Q+1) + (Q+1-D)s + c_1 D + \tilde{s}\sum_{N=0}^{D-1}\left(\begin{array}{c} Q+1 \\ N \end{array}\right)\alpha^N\,(1-\alpha)^{Q+1-N}\,[D-N]$$

or, simplifying the notation:

$$f\,(Q+1) = u(Q+1) + (Q+1-D)s + c_1 D + \tilde{s}\sum_{N<D}[D-N]p\,[N|Q+1] \tag{15}$$

From probability theory we have:

$$p\,(N|Q+1)) = \alpha p\,(N-1|Q) + (1-\alpha)\,p\,(N|Q) \tag{16}$$

Subtracting (8) from (15) and using (16) gives us the following expression for the first difference:

$$\Delta f\,(Q) = u + s + \tilde{s}\alpha\sum_{N<D}[D-N]\,\{p\,[N-1|Q] - p\,[N|Q]\}$$

which simplifies to:

$$\Delta f\,(Q) = u + s - \tilde{s}\alpha p\,(N<D|Q)$$

Letting $\Psi(Q) = p(N \geq D|Q)$:

$$\Delta f(Q) = u + s - \tilde{s}\alpha + \tilde{s}\alpha(\Psi(Q)) \tag{17}$$

Since $\Psi(Q)$ is strictly increasing for all $Q \geq D$, (17) is a monotonically increasing function of Q. We conclude the proof by noting that, since $\lim_{Q\to\infty} \Delta f(Q) = u + s > 0$, (17) has a unique minimum.

Proof of Proposition 4

Assume that $\frac{D}{\alpha}$ items are acquired. Note that acquiring one fewer item reduces costs when the following condition holds:

$$u > \tilde{s}\left[\sum_{N=0}^{D-1}\binom{Q-1}{N}\alpha^N(1-\alpha)^{Q+1-N}(D-N) - \sum_{N=0}^{D-1}\binom{Q}{N}\alpha^N(1-\alpha)^{Q+1-N}(D-N)\right] \tag{18}$$

That is, when the unit acquisition cost saved exceeds the expected increase in costs. When (18) does not hold, acquiring one fewer item does not reduce costs. From probability theory we have:

$$p(N|Q) = \alpha p(N-1|Q-1) + (1-\alpha)p(N|Q-1) \tag{19}$$

Simplifying (18) using (19):

$$u > \tilde{s}\alpha \sum_{N<D}(D-N)\left[p(N-1|Q-1) - p(N|Q-1)\right] \tag{20}$$

which simplifies to:

$$u > \tilde{s}\alpha p(N < D|Q-1) \tag{21}$$

Note that, for sufficiently large D and Q, the expected number of low cost items will be symmetrically distributed with mean αQ. Thus, when $Q = \frac{D}{\alpha}$, the likelihood of a shortage of low cost items is 0.5. Given (21), we know that $p(N < D|Q-1) > p(N < D|Q) = 0.5$, so $p(N < D|Q-1) = 0.5 + \epsilon$ and (21) can be written $u > \tilde{s}\alpha(0.5 + \epsilon)$. Therefore, when $u \leq \frac{\alpha\tilde{s}}{2}$, then $u \leq \frac{\alpha\tilde{s}}{2} < \tilde{s}\alpha p(N < D|Q-1)$ and (21) does not hold, so decreasing Q will not reduce costs. In addition, for sufficiently large D and Q, ϵ will be sufficiently small such that $u > \frac{\alpha\tilde{s}}{2}$ implies $u > \tilde{s}\alpha p\,(N < D|Q-1)$, and decreasing Q by one unit reduces expected cost.

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